

Optimal Low Energy Earth-Moon Transfers

Paul Ricord Griesemer¹ and Cesar Ocampo²
The University of Texas at Austin, Austin, Texas, 78712

and
D. S. Cooley³
NASA Goddard Space Flight Center, Greenbelt, Maryland, 20771

The optimality of a low-energy Earth-Moon transfer is examined for the first time using primer vector theory. An optimal control problem is formed with the following free variables: the location, time, and magnitude of the transfer insertion burn, and the transfer time. A constraint is placed on the initial state of the spacecraft to bind it to a given initial orbit around a first body, and on the final state of the spacecraft to limit its Keplerian energy with respect to a second body. Optimal transfers in the system are shown to meet certain conditions placed on the primer vector and its time derivative. A two point boundary value problem containing these necessary conditions is created for use in targeting optimal transfers. The two point boundary value problem is then applied to the ballistic lunar capture problem, and an optimal trajectory is shown. Additionally, the ballistic lunar capture trajectory is examined to determine whether one or more additional impulses may improve on the cost of the transfer.

Nomenclature

WSB	=	weak stability boundary
J	=	performance index of the optimal control problem
t	=	time
$\Delta \mathbf{v}$	=	change of velocity vector relating to an impulsive maneuver
$\delta(t)$	=	impulse function
$\delta_n(t-t_c)$	=	delta sequence
a_{th}	=	thrust magnitude
\dot{m}	=	mass flow rate of the engine propellant
c	=	characteristic velocity of the engine propellant
m	=	mass of the spacecraft
τ	=	time-like variable indicating the position of the spacecraft on a given orbit
\mathbf{u}	=	vector of controls
C	=	control constraint
\mathbf{r}	=	radius vector
\mathbf{v}	=	velocity vector
\mathbf{g}	=	acceleration vector of the general force field
Θ	=	boundary conditions at the initial time
m^*	=	pre-defined initial mass of the spacecraft
E^*	=	desired Keplerian energy state
E	=	actual Keplerian energy state

¹ Graduate Student, Department of Aerospace Engineering and Engineering Mechanics, 1 University Station C0600, The University of Texas, Austin, TX 78712.

² Associate Professor, Department of Aerospace Engineering and Engineering Mechanics, 1 University Station C0600, The University of Texas, Austin, TX 78712.

³ Aerospace Engineer, Navigation and Mission Design Branch, NASA Goddard Space Flight Center, Mailstop 595.0, Greenbelt, MD 20771.

ψ	=	boundary condition at the final time
α	=	slack variable
G	=	endpoint function
v	=	constant Lagrange multiplier associated with the final boundary condition
ξ	=	vector of constant Lagrange multipliers associated with the initial boundary conditions
μ	=	Lagrange multiplier associated with the control constraint
\hat{H}	=	extended Hamiltonian function
λ	=	adjoint variable vector
\mathbf{p}	=	primer vector
\mathbf{f}	=	state equation vector
\mathbf{a}	=	optimization parameters in the numerical optimization problem
\mathbf{c}	=	vector of constraints in the numerical optimization problem
$\Phi(t_f, t_0)$	=	state transition matrix
\mathbf{X}	=	vector of state variables
a	=	semi-major axis
e	=	eccentricity
i	=	inclination
ω	=	argument of perigee
Ω	=	right ascension of the ascending node

Introduction

With Belbruno's introduction of the Weak Stability Boundary (WSB),¹ and Belbruno and Miller's² subsequent success in employing its concepts in the Japanese Hiten lunar mission, low energy transfers have been shown to be a useful tool in transferring between orbits around different gravitational bodies. Although they have demonstrated savings in fuel consumption when compared to other classes of transfers, including Hohmann transfers, the question of finding optimal WSB transfers raises issues not previously explored. This work proposes a method of designing optimal impulsive low energy transfers from a given orbit around a primary body to a captured state with respect to a secondary body. The method relies on new optimality conditions for a ballistically captured spacecraft that are derived from primer vector theory. The new conditions are used in a numerical targeting algorithm to find optimal WSB transfers between the Earth and the Moon.

The problem of minimizing the fuel consumption associated with impulsive orbit-to-orbit transfers has received a great amount of attention. Formalized by Lawden in 1963,³ the necessary conditions for an optimal impulsive transfer are imposed on the *primer vector*, the reflection of the vector of adjoint variables in the optimal control problem that correspond to the velocity of the spacecraft. Lawden's necessary conditions specify the magnitude and orientation of the primer vector at the times of impulse. Specifically, the primer vector must align with the direction of the impulse and have a uniform magnitude at each impulse. The magnitude has been traditionally chosen to be unity at each impulse, but the designated value is arbitrary. Lion and Handelsman⁴ extended Lawden's work to determine when an additional impulse would improve on the fuel cost of the transfer. Jezewski⁵ later gives a

summary of primer vector theory, and introduces an estimation of the magnitude of an intermediate impulse that would optimize the transfer.

Primer vector theory has been used to evaluate the optimality of several different types of transfers. Optimal transfers in an inverse square gravitational field have been analyzed for various types of missions. Rendezvous in the vicinity of a circular orbit has been explored by Prussing,⁶ direct ascent rendezvous by Gross and Prussing,⁷ and rendezvous between circular orbits by Prussing and Chiu.⁸ Jezewski⁵ has studied the problem of multiple impulse transfers and transfers with inequality constraints. The primer vector in the three body problem was studied by Hiday-Johnston and Howell⁹ to evaluate the optimality of transfers between libration point orbits, by D'Amario and Edelbaum¹⁰ in the transfer from a collinear libration point to a lunar orbit, and by Ocampo¹¹ in the problem of insertion into distant retrograde orbits.

The use of primer vector theory in the four body problem, which is required for Lunar WSB transfers, has been less comprehensive. In one example, Pu and Edelbaum¹² created a numerical method to compute 2-impulse and 3-impulse optimal transfers in the four body problem. Once the trajectories were numerically optimized, the primer vector history of the transfers was evaluated to verify their optimality.

Previous implementations of primer vector theory have been insufficient to deal with the lunar WSB transfer due to its ballistic nature. Necessary conditions have been traditionally based on the magnitude and orientation of the primer vector at the times of impulse. In a ballistic capture scenario with only one impulse occurring during the entire transfer, necessary conditions relating to the impulse provide an insufficient number of conditions to determine an optimal transfer. New conditions must be derived specifically for a ballistic capture to produce the required number of necessary conditions to define local optimality.

The goal of the current research is to extend the previous work by designing optimal trajectories that transfer from a specified orbit around a body to an unspecified elliptical orbit around a second body with a single impulse. Primer vector theory will be extended to deal with the orbit-to-capture problem and used to target optimal trajectories. A particular transfer in the Sun-Earth-Moon system will be found using the targeting algorithm and then analyzed to determine whether the transfer can be improved with an intermediate impulse.

Primer Vector Theory for an Impulsive Low Energy Transfer

Consider the orbit-to-capture transfer problem for an impulsively maneuvering spacecraft in a general force field. A spacecraft orbiting a body is to be transferred to any captured orbit of another body via a single impulse. In this situation, capture is defined by a negative Keplerian energy level with respect to the second body, assuring an instantaneous elliptical orbit around the body. The derivation of the necessary conditions for an optimal control of this spacecraft is based on Hull's¹³ formulation of the optimal control problem. Where the derivation overlaps with the orbit-to-orbit problem, it has equivalent results to those of Lawden, Jezewski, and Lion and Handelsman.^{3,4,5}

Cost function

An optimal control will minimize the total impulse provided to the spacecraft, represented by the following performance index,

$$J = \int_{t_1}^{t_2} \Delta v_1 \delta(t - t_1) dt, \quad (1)$$

where Δv_1 is the magnitude of the initial impulses, t_2 is defined to be the final time of the transfer, and $\delta(t - t_1)$ is an impulse function in which the impulse occurs at the beginning of the transfer, $t = t_1$. The impulse function is used as a way to represent a finite change in the velocity of the spacecraft due to an infinite acceleration. By containing the limit functions within the impulse function, the necessary conditions for optimal control can be derived without taking the limit of the performance index itself. The delta function used here is defined by the relationship

$$\delta(t - t_c) = \lim_{n \rightarrow \infty} \delta_n(t - t_c), \quad (2)$$

where $\delta_n(t - t_c)$ is a delta sequence that may be defined by the following piecewise continuous function:¹⁴

$$\delta_n(t - t_c) = \begin{cases} 0 & \text{for } t < t_c - \frac{1}{2n} \\ n & \text{for } t_c - \frac{1}{2n} < t < t_c + \frac{1}{2n} \\ 0 & \text{for } t_c + \frac{1}{2n} < t \end{cases} \quad (3)$$

When used in the cost function, the delta function represents an impulsive maneuver. The velocity change and fuel consumption associated with the maneuver can be determined through the integration of the function. The integral of the delta function has the property¹⁴

$$\int_{-\infty}^{\infty} f(t) \delta(t - t_c) dt = f(t_c) . \quad (4)$$

The resulting velocity change due to a maneuver at time c can be determined explicitly using Eq. (4).

$$\begin{aligned} \Delta \mathbf{V}_c &= \int_{t_c^-}^{t_c^+} \Delta v_c \delta(t - t_c) \mathbf{l}_c dt \\ &= \int_{-\infty}^{\infty} \Delta v_c \delta(t - t_c) \mathbf{l}_c dt \\ &= \Delta v_c \mathbf{l}_c \end{aligned} \quad (5)$$

In this expression, \mathbf{l}_c is the steering vector that orients the impulse. The mass consumed by a maneuver can be calculated from the mass state equation. If thrust acceleration, a_{th} , is represented by the expression

$$a_{th} = \frac{\dot{m}c}{m} \quad (6)$$

where \dot{m} is the mass flow rate of the propellant and c is its characteristic velocity, then the following equation holds for an impulsive maneuver occurring at time t_1 :

$$\Delta v_1 \delta(t - t_1) = \frac{\dot{m}c}{m} \quad (7)$$

or

$$\dot{m} = -\frac{m\Delta v_1}{c} \delta(t - t_1) . \quad (8)$$

Optimal control problem

The controls in the orbit-to-capture problem, shown in Eq. (9), are the magnitude and direction of the transfer insertion maneuver at time t_1 and the location of the spacecraft on a specified orbit around a gravitational body, represented by the time-like parameter τ_1 , discussed below. Both t_1 and t_2 are free parameters.

$$\mathbf{u} = (\Delta v_1 \quad \mathbf{l}_1^T \quad \tau_1)^T \quad (9)$$

A control constraint exists on the thrust direction vector \mathbf{l}_1 so that its magnitude is unity.

$$C = (\mathbf{l}_1^T \mathbf{l}_1 - 1) = 0 \quad (10)$$

The differential constraint for the problem is the state equation,

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{g} + \delta(t - t_1) \Delta v_1 \mathbf{l}_1 \\ -\frac{m \Delta v_1}{c} \delta(t - t_1) \end{pmatrix}, \quad (11)$$

where \mathbf{r} , \mathbf{v} , and m , are the position vector, the velocity vector, and the mass of the spacecraft, respectively. The gravitational acceleration vector \mathbf{g} represents a general force field, and although the low energy lunar transfer is typically calculated with a four body system, no assumptions about the composition of the gravity field will be made in the derivation. The acceleration vector for the spacecraft consists of the gravitational acceleration and the acceleration due to the impulsive maneuver.

The boundary conditions are such that they constrain the initial state of the spacecraft to lie on a pre-established orbit. Following the methodology of Ocampo,¹¹ a time-like variable, τ_1 is introduced into the problem to define a plane in phase space that contains the initial orbit. This variable is defined such that $\mathbf{r}(\tau_1)$ and $\mathbf{v}(\tau_1)$ lie on the initial orbit for any τ_1 . In this way, τ_1 is similar to the true anomaly of an orbit, or any other variable that defines a spacecraft's location on a particular orbit. The boundary conditions that constrain the initial state of the spacecraft have the following form:

$$\boldsymbol{\theta} = \begin{pmatrix} \mathbf{r}_1 - \mathbf{r}(\tau_1) \\ \mathbf{v}_1 - \mathbf{v}(\tau_1) \\ m_1 - m_1^* \end{pmatrix} = \mathbf{0}, \quad (12)$$

where m_1^* is the pre-defined initial mass of the spacecraft.

The boundary conditions at the final time are generalized to require the spacecraft to arrive at the second body with a specific energy state with respect to the second body less than or equal to a stated value. For a spacecraft to be captured in a two body system, the stated energy level would be less than zero. In a multi-body system, however, a stronger constraint may be desired to increase the likelihood of the spacecraft orbiting the second body before it escapes. In the derivation, the desired energy level will be left unspecified. An inequality constraint is used to allow

for the possibility of an optimal transfer arriving at the secondary body at a more strongly captured state than specified. The resulting boundary condition at the final time is

$$\psi = E^* - E \leq 0, \quad (13)$$

where E^* is the desired energy state and E is the energy of the spacecraft with respect to the Moon as calculated by the vis-viva equation:

$$E = \frac{v_s^2}{2} - \frac{\mu_s}{r_s}, \quad (14)$$

where r_s and v_s are the radius and velocity with respect to the secondary body, and μ_s is the gravitational parameter of the secondary body.

Following the methodology of Hull,¹³ a slack variable is introduced to convert the inequality constraint in Eq. (13) into an equality constraint. A consequence of the introduction of a slack variable into the constraint is an additional parameter in the problem. The boundary condition becomes

$$\psi = \frac{v_s^2}{2} - \frac{\mu_s}{r_s} - E^* + \alpha^2 = 0 \quad (15)$$

where α is the slack variable.

The constraints of the problem define the endpoint function to be

$$G' = v\psi + \xi^T \boldsymbol{\theta} \quad (16)$$

where v and ξ are a scalar and a vector, respectively, of constant Lagrange multipliers adjoined to the endpoint constraints.

An extended Hamiltonian is formed by using a Lagrange multiplier, μ , to adjoin the control constraint in Eq. (10) to the Hamiltonian,

$$\hat{H} = \boldsymbol{\lambda}_r^T \mathbf{v} + \boldsymbol{\lambda}_v^T (\mathbf{g} + \Delta v_1 \delta(t - t_1) \mathbf{I}_1) - \lambda_m \left(\frac{m \Delta v_1}{c} \delta(t - t_1) \right) + \Delta v_1 \delta(t - t_1) + \mu C. \quad (17)$$

The extended Hamiltonian is constant if the gravitational acceleration vector is not a function of time. It is not necessarily an integral of the problem for the general force field.

The performance index and constraints described above form a time-free optimal control problem. The necessary conditions from the first differential are given by the Euler-Lagrange equations

$$\dot{\mathbf{x}} = \mathbf{f} \quad (18)$$

$$\dot{\boldsymbol{\lambda}} = -\hat{H}_{\mathbf{x}}^T \quad (19)$$

$$0 = \hat{H}_{\mathbf{u}}^T \quad (20)$$

and the natural boundary conditions

$$\begin{aligned} \hat{H}_2 &= -G'_{t_2} \\ \hat{H}_1 &= G'_{t_1} \\ \boldsymbol{\lambda}_2 &= G'^T_{\mathbf{x}_2} \\ \boldsymbol{\lambda}_1 &= -G'^T_{\mathbf{x}_1} \end{aligned} \quad (21)$$

The conditions of optimality pertaining to the impulse in the orbit-to-capture problem are equivalent to the orbit-to-orbit necessary conditions derived in the work of Lawden, Jezewski, and others. As previously stated, the primer vector must be aligned with the impulse at the time of the impulse, the time derivative of its magnitude must be zero, and its magnitude must be unity:

$$\mathbf{p}_1 = \frac{\Delta \mathbf{V}_1}{\Delta v_1} \quad (22)$$

$$\dot{p}_1 = 0, \quad (23)$$

where

$$\mathbf{p} = -\boldsymbol{\lambda}_v \quad (24)$$

At time t_2 , there are two complications that call for necessary conditions that differ from the conventional derivation. First, the inequality endpoint constraint does not fit neatly into the optimal control theory formulation of Hull. After extending Hull's formulation to account for an endpoint inequality constraint, the inclusion of the slack variable produces two separate cases, one when the solution is on the boundary of the constraint, and one when it is off of the boundary. Secondly, the traditional orbit-to-orbit problem contains at least two impulses. A first impulse

initiates the transfer, and a second inserts the spacecraft from a transfer trajectory into the target orbit. This paradigm is insufficient when dealing with ballistic capture problems. New conditions need to be derived from primer vector theory to account for the possibility of reaching the desired state without an impulse to conclude the transfer.

Hull's derivation allows for the endpoint function to depend on the initial and final time, the state variables, and constant Lagrange multipliers. The slack variable associated with the inequality constraint cannot be considered to belong to any class of these variables. It can be dealt with outside of Hull's formulation by treating it as a separate class of variable and including it in a modified endpoint function, G' . The modified endpoint function shown in Eq. (16) depends on the initial and final time, the state variables, constant Lagrange multipliers, and the slack variable, α .

To determine necessary conditions appropriate for an orbit to capture trajectory at the terminal time, consider the endpoint conditions on λ from Eq. (21):

$$\begin{aligned}\lambda_{r_2} &= G'_{r_2}{}^T = v \frac{\mu_s}{r_s^3} \mathbf{r}_s \\ \lambda_{v_2} &= G'_{v_2}{}^T = v v_s \mathbf{v}_s\end{aligned}\quad (25)$$

Equation (25) establishes new necessary conditions for the orbit-to-capture problem. It dictates that the primer vector at the final time is aligned with the velocity vector of the spacecraft with respect to the second gravitational body, and the rate of change of the primer vector is aligned with the radius vector of the spacecraft with respect to the gravitational body. Furthermore, because the constant Lagrange multiplier v appears in both expressions, its value is not arbitrary. The parameter v establishes a relationship between the magnitude of the primer vector and its time derivative.

Hull derives the first order conditions of optimality by taking the first differential of the extended cost function, which includes the endpoint function. By adding the slack variable to the parameters of the problem, a term appears in the differential that depends on $d\alpha$,

$$\frac{dG'}{d\alpha} = 2v\alpha\quad (26)$$

In order for a trajectory to be optimal, the expression in Eq. (26) must be equal to zero. This optimality condition can be satisfied in two cases. In the first case, α is equal to zero and the solution lies on the boundary of the inequality constraint. In the second case the Lagrange multiplier, v , is equal to zero. This condition applies to the

situation where the optimality conditions are satisfied on a trajectory that terminates with an energy level less than the targeted energy level. When this condition is applied to the previous optimality condition in Eq. (25), it can be seen that the primer vector and the derivative of its magnitude must be zero. All other conditions on G' are equivalent to the conditions on G in Hull's formulation.

Together, Eqns. (25) and (26) define the necessary conditions at the final time for a ballistic capture trajectory. They apply to the magnitude and direction of both the primer vector and the time derivative of the primer vector, and represent trajectories that terminate with an energy level either equal to that targeted or less than the targeted value. Additionally, it can be shown that the magnitude of the primer vector must be less than unity to preclude an additional impulse.

Additional Impulses

A separate analysis can be made to determine whether an intermediate impulse has the potential to reduce the cost of the maneuver. Consider a trajectory that has an additional maneuver at time t_k . The performance index of the trajectory would be changed to

$$J = \int_{t_1^-}^{t_2^+} \Delta v_1 \delta(t - t_1) + \Delta v_k \delta(t - t_k) dt, \quad (27)$$

and the vector of controls would include the magnitude, time, and direction of the intermediate maneuver,

$$\mathbf{u} = (\Delta v_1 \quad \mathbf{I}_1 \quad \tau_1 \quad \Delta v_k \quad \mathbf{I}_k \quad t_k)^T. \quad (28)$$

The unit thrust vectors are subject to the constraint

$$\mathbf{C} = \begin{pmatrix} \mathbf{I}_1^T \mathbf{I}_1 - 1 \\ \mathbf{I}_k^T \mathbf{I}_k - 1 \end{pmatrix} = \mathbf{0}, \quad (29)$$

and the extended Hamiltonian is

$$\begin{aligned} \hat{H} = & \lambda_r^T \mathbf{v} + \lambda_v^T (\mathbf{g} + \Delta v_1 \delta(t - t_1) \mathbf{I}_1 + \Delta v_k \delta(t - t_k) \mathbf{I}_k) \\ & - \lambda_m \left(\frac{m \Delta v_1}{c} \delta(t - t_1) + \frac{m \Delta v_k}{c} \delta(t - t_k) \right) + \Delta v_1 \delta(t - t_1) + \Delta v_k \delta(t - t_k) + \boldsymbol{\mu}^T \mathbf{C}. \end{aligned} \quad (30)$$

Applying the optimality condition in Eq. (20) to Eq. (30) gives the following result when considering the magnitude of the primer vector at an intermediate impulse:

$$\hat{H}_{\Delta v_k} = \delta(t - t_k) \lambda_v^T \mathbf{I}_k - \lambda_m \frac{m}{c} \delta(t - t_k) + \delta(t - t_k) = 0 \quad (31)$$

At time t_k Eq. (31) reduces to

$$|\lambda_{v_k}| = \lambda_{m_1} \frac{m_1}{c} - 1, \quad (32)$$

where

$$m_k = m_1 \left(1 - \frac{\Delta v_1}{c} \right) \quad (33)$$

and

$$\lambda_{m_k} = \lambda_{m_1} \left(1 + \frac{\Delta v_1}{c} \right). \quad (34)$$

The result obtained from applying the same condition at the initial time is

$$|\lambda_{v_1}| = \lambda_{m_1} \frac{m_1}{c} - 1. \quad (35)$$

It can be seen from comparing Eq. (32) to Eq. (35) that a necessary condition for an intermediate impulse is

$$|\mathbf{p}_k| = |\mathbf{p}_1|. \quad (36)$$

Equation (36) requires that an optimal trajectory with an intermediate impulse have primer vectors with equal magnitudes at each impulse. As stated previously, it is common to assign a unit magnitude to the primer vector at the initial impulse. Consequently, if the primer vector never again returns to unit magnitude between times t_1 and t_2 , there is no potential for an improvement in the performance index via an intermediate impulse.

Numerical Analysis

The necessary conditions on the state and co-state vectors described above can be used in a targeting algorithm to optimize near-optimal trajectories. It should be stressed that convergence is an issue when working with these chaotic trajectories in the four body problem. A reasonable first guess is required to converge upon an optimal solution, and convergence is not guaranteed even in this case. A particularly problematic case for convergence is the

case of a lunar flyby on the outbound portion of the transfer. Although transfers with these flybys can be more efficient, they will not be considered here due to the numerical problems.

The boundary conditions and the necessary conditions of the optimal control problem define a two point boundary value problem that can be numerically solved to yield a control that satisfies the conditions of optimality. The unknown and constraint vectors associated with the problem are shown in Eq. (37). The differential equations to be satisfied are the state and co-state equations shown in Eqns. (18) and (19).

$$\begin{aligned} \mathbf{a}_{11 \times 1} &= (t_1 \quad t_2 \quad \tau_1 \quad \dot{\mathbf{p}}_1 \quad \Delta \mathbf{v}_1 \quad v \quad \alpha)^T, \\ \mathbf{c}_{11 \times 1} &= \left(\hat{H}_1 \quad \hat{H}_2 \quad E - E^* + \alpha^2 \quad 2\alpha\mu \quad \lambda_{r_2} - v \frac{\mu_s}{r_s^3} \mathbf{r}_s \quad \lambda_{v_2} - v v_s \mathbf{v}_s \quad \dot{p}_1 \right)^T \end{aligned} \quad (37)$$

This system, with an appropriate initial guess, can be solved by a numerical nonlinear equation solver such as NS-12.¹⁵ The initial state vector is formed from the prescribed initial orbit and the parameter τ_1 . The initial co-state vector is determined from the parameter $\dot{\mathbf{p}}_1$ and by aligning the primer vector in the direction of the impulse and giving it a magnitude of unity.

A solution to this system is not guaranteed to meet the necessary conditions of optimality, as there are no path constraints to ensure the magnitude of the primer vector remains below unity. An inspection of the primer vector history to verify this condition is required to ensure the trajectory meets the necessary conditions once a solution to the boundary value problem is found.

The Restricted Four Body Problem

The two point boundary value problem shown above can be solved for any dynamical system. However, single impulse ballistic lunar transfer trajectories rely on the gravitational attraction of the Sun, Earth, and Moon.² In this paper, the governing dynamics of the spacecraft are generated by assuming these three primary bodies act as point masses attracting the spacecraft. In the numerical analysis, their orbits about each other are defined by the JPL DE405 ephemeris.¹⁶ A spacecraft moving through their gravitational field is assumed to be infinitesimally small, so that it will not perturb the orbits of the primary masses. A non-rotating coordinate system is established that is centered on the Earth-Moon barycenter. Assuming Newtonian gravity, the gravitational acceleration of a spacecraft in this four-body system is defined in Eq. (38).¹⁷

$$\ddot{\mathbf{r}} = -\frac{\mu_s}{r_s^3}\mathbf{r}_s - \frac{\mu_e}{r_e^3}\mathbf{r}_e - \frac{\mu_m}{r_m^3}\mathbf{r}_m - \frac{\mu_s + \mu_e + \mu_m}{r_{s/o}^3}\mathbf{r}_{s/o} \quad (38)$$

The terms \mathbf{r}_s , \mathbf{r}_e , and \mathbf{r}_m refer to the position vectors of the spacecraft with respect to the Sun, Earth, and Moon, respectively, and $\mathbf{r}_{s/o}$ refers to the position of the Sun with respect to the origin of the coordinate frame. The gravitational parameters μ_s , μ_e , and μ_m are defined in Table 1. Equation (38) contains three direct gravitational accelerations followed by an indirect term. The indirect term accounts for the acceleration of the reference frame around the Sun-Earth-Moon barycenter.

Table 1 Gravitational Parameter Values

<u>Gravitational Parameter</u>	<u>Value (km³/sec²)</u>
μ_s	1.32715×10^{11}
μ_e	3.986004×10^5
μ_m	4.9029×10^3

State Transition Matrix Derivatives

As documented by Zimmer and Ocampo,¹⁸ the use of analytical derivatives in targeting algorithms that require gradients increases both the speed and likelihood of convergence. The algorithm used to target optimal trajectories requires partial derivatives that relate the constraints, \mathbf{c} , of the problem to the free parameters, \mathbf{a} . For the constraints that are applied at the initial time, partial derivatives can be derived directly from the initial state. For the constraints at the final time, the state transition matrix can be used in conjunction with the chain rule to determine partial derivatives that relate parameters at the initial time to the final constraints. It should be noted, however, that the state transition matrix can only relate changes in the state that occur at a common time. Therefore, when relating constraint values to the variation of the initial time, an additional step must be taken. The changes in the state at $t = t_1$ must be related to the change in the initial time through calculating a numerical derivative that relates to change in the state at a fixed time t_1 to a change in the initial time. This derivative can be calculated with the forward difference method by simply integrating the state from a perturbed time back to t_1 . The partial derivatives used in the numerical routine are shown below.

$$\frac{\partial \mathbf{X}(t_2)}{\partial t_1} = \Phi(t_2, t_1) \frac{\partial \mathbf{X}(t_1)}{\partial t_1} \quad (39)$$

where $\partial[x(t_1)]/\partial t_1$ is calculated using the forward difference method by varying the initial time by a small amount

and numerically integrating back to t_1 .

$$\frac{\partial c}{\partial a} = \begin{pmatrix} \lambda_v^\top \frac{\partial \mathbf{g}_1}{\partial t_1} & 0 & \frac{\partial \hat{H}_1}{\partial \mathbf{X}_1} \frac{\partial \mathbf{X}_1}{\partial f} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \frac{\partial \hat{H}_2}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial t_1} & \frac{\partial \hat{H}_2}{\partial \mathbf{X}_2} \frac{\partial \mathbf{X}_2}{\partial t_2} & \frac{\partial \hat{H}_2}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial f} & \frac{\partial \hat{H}_2}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \mathbf{p}_1} & \frac{\partial \hat{H}_2}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \Delta \mathbf{v}_1} & 0 & 0 \\ \frac{\partial c_3}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial t_1} & \frac{\partial c_3}{\partial \mathbf{X}_2} \frac{\partial \mathbf{X}_2}{\partial t_2} & \frac{\partial c_3}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial f} & \frac{\partial c_3}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \mathbf{p}_1} & \frac{\partial c_3}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \Delta \mathbf{v}_1} & 0 & -2\alpha \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 2\mu \\ \frac{\partial c_5}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial t_1} & \frac{\partial c_5}{\partial \mathbf{X}_2} \frac{\partial \mathbf{X}_2}{\partial t_2} & \frac{\partial c_5}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial f} & \frac{\partial c_5}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \mathbf{p}_1} & \frac{\partial c_5}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \Delta \mathbf{v}_1} & -\frac{\mu_s}{r_s^3} \mathbf{r}_s & 0 \\ \frac{\partial c_6}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial t_1} & \frac{\partial c_6}{\partial \mathbf{X}_2} \frac{\partial \mathbf{X}_2}{\partial t_2} & \frac{\partial c_6}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial f} & \frac{\partial c_6}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \mathbf{p}_1} & \frac{\partial c_6}{\partial \mathbf{X}_2} \Phi(t_2, t_1) \frac{\partial \mathbf{X}_1}{\partial \Delta \mathbf{v}_1} & -v_s \mathbf{v}_s & 0 \\ 0 & 0 & 0 & \frac{1}{\dot{p}_1} \dot{\mathbf{p}}_1^\top & \mathbf{0} & 0 & 0 \end{pmatrix} \quad (40)$$

where

$$\frac{\partial \hat{H}_2}{\partial \mathbf{X}_2} = \begin{pmatrix} \lambda_{v_2}^\top \frac{\partial \mathbf{g}_2}{\partial \mathbf{r}_2} & \lambda_{r_2}^\top & \mathbf{g}_2^\top & \mathbf{v}_2^\top \end{pmatrix} \quad (41)$$

$$\frac{\partial (E - E^* + \alpha^2)}{\partial \mathbf{X}_2} = \begin{pmatrix} \frac{\mu_m}{r_m^3} \mathbf{r}_m^\top & \mathbf{v}_m^\top & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (42)$$

$$\frac{\partial c_5}{\partial \mathbf{X}_2} = \begin{pmatrix} v \mu_m \left(-\frac{1}{r_m^3} \mathbf{I} + 3 \frac{1}{r_m^5} \mathbf{r}_m \mathbf{r}_m^\top \right) & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \quad (43)$$

$$\frac{\partial c_6}{\partial \mathbf{X}_2} = \begin{pmatrix} \mathbf{0} & -v \left(v_m \mathbf{I} + \frac{1}{v_m} \mathbf{v}_m \mathbf{v}_m^\top \right) & \mathbf{I} & \mathbf{0} \end{pmatrix} \quad (44)$$

$$\frac{\partial \mathbf{X}_2}{\partial t_2} = \begin{pmatrix} \mathbf{v}^\top & \mathbf{g}^\top & \dot{\lambda}_r^\top & \dot{\lambda}_v^\top \end{pmatrix} \quad (45)$$

$$\frac{\partial \mathbf{X}_1}{\partial \mathbf{p}_1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix} \quad (46)$$

$$\frac{\partial \mathbf{X}_1}{\partial \Delta \mathbf{v}_1} = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (47)$$

$$\frac{\partial \mathbf{X}_1}{\partial f} = \begin{pmatrix} a(-\cos \Omega \sin f + \sin \Omega \cos f \cos i) \\ a(-\sin \Omega \sin f + \cos \Omega \cos f \cos i) \\ a \cos f \sin i \\ -v(\cos \Omega \cos f - \sin \Omega \sin f \cos i) \\ -v(\sin \Omega \cos f + \cos \Omega \sin f \cos i) \\ -v \sin f \sin i \end{pmatrix} \quad (48)$$

Note that in these equations the initial orbit around the Earth is circular, where a is the semi-major axis of the orbit, f is the true anomaly of the orbit, i is the inclination, v is the magnitude of the velocity, and Ω is the right ascension of the ascending node. $\Phi(t_2, t_1)$ is the state transition matrix associated with the four body equations of motion given in Eq. (38). These partial derivatives are employed in the targeting code to iterate from an initial guess to an optimal solution. The guesses are propagated using a numerical predictor-corrector integration method.¹⁹

Results

An optimal transfer is sought from the low Earth orbit and detailed in Table 2 to a captured state around the Moon using a single impulsive maneuver. A previously targeted non-optimal transfer is used as an initial guess, and improved upon by targeting the necessary conditions developed above. Fig. 1 shows a converged solution in non-rotating coordinates centered at the Earth-Moon barycenter.

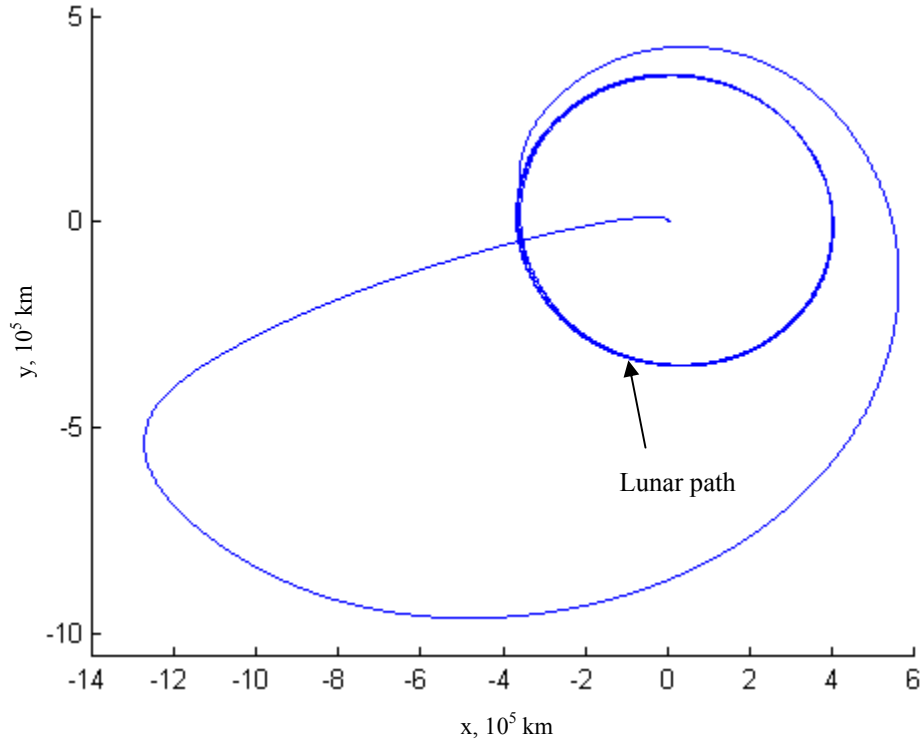
Table 2 Parking orbit properties at t_0

<u>Orbit property</u>	<u>Value</u>
epoch (JED)	2455658.47677
a (km)	7200
e	0
i (deg)	47.188
ω (deg)	0
Ω (deg)	1.09948

In an effort to find a suitable initial guess for the targeting of an optimal transfer, parameters from a non-optimal ballistic lunar capture trajectory were calculated using a method detailed in previous work.²⁰ The parameters of the transfer are listed in Table 3, where the subscript (v) refers to the component of the vector in the direction of the velocity, the subscript (h) refers to the component in the direction of the angular momentum, and the subscript (r) refers to the direction that completes the right-handed coordinate system. The final state of the transfer lies on an elliptical two body orbit about the Moon.

Table 3 Transfer properties

Transfer Property	Value
ΔV_v (km/sec)	3.05047
ΔV_r (km/sec)	0.0
ΔV_h (km/sec)	0.0
Δt (days)	117.32581
f (deg)	22.6881

**Fig. 1 Single burn low energy transfer to lunar orbit**

The parameters of this transfer are used as an initial guess in the two point boundary value problem described above. The initial values of the co-state associated with the velocity are assigned to align the primer vector with the impulse and give it a unit magnitude. The remaining initial values, those in the co-state associated with the radius, are assigned very small numbers.

The solution to the two point boundary value problem in Eq. (37) yields a transfer that meets all of the necessary conditions of optimality. The time history of the magnitude of the primer vector for the converged solution to the two point boundary value problem is shown in Fig. 2. This primer vector history satisfies the necessary conditions detailed above: it begins with a magnitude of unity and a slope of zero, shown in detail in Fig. 3, and never again returns to a magnitude of unity. Additionally, this solution lies on the boundary of the inequality constraint, with

slack variable, α , having a value of zero for this transfer and satisfying the condition in Eq. (26). Finally, it is not evident in Fig. 1 Single burn low energy transfer to lunar orbit, but the primer vector at the final time is aligned with the velocity vector and its time derivative is aligned with the radius vector with respect to the moon. Not only are the vectors aligned but their magnitudes are related through the Lagrange multiplier v . Table 4 shows the angles of the related vectors relative to their counterparts.

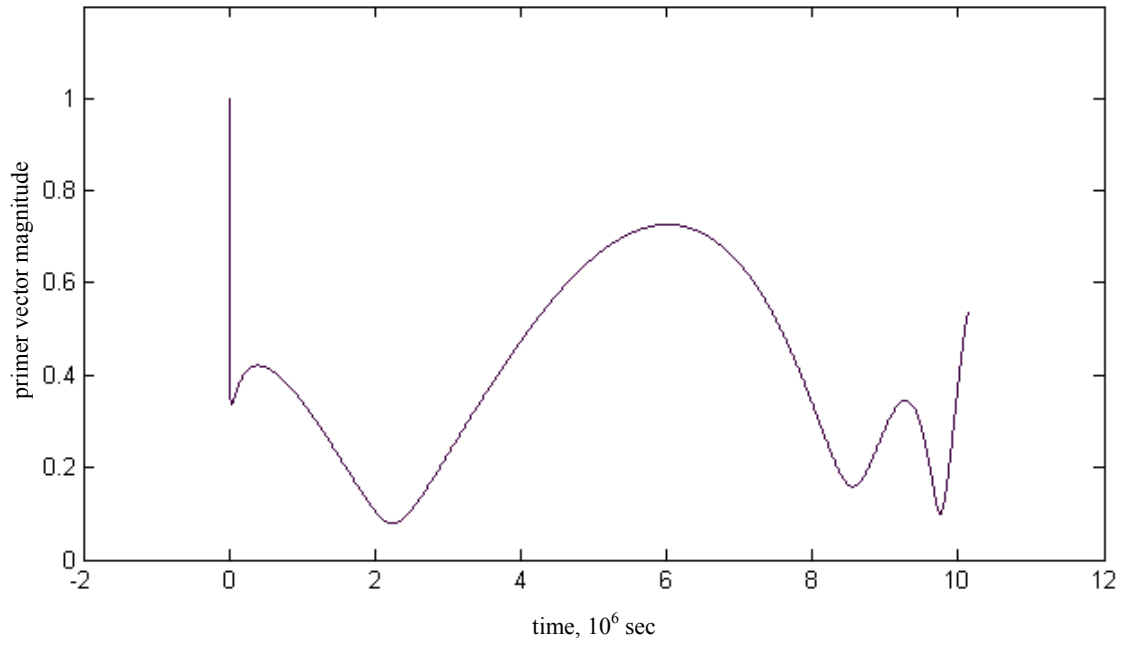


Fig. 2 Primer vector magnitude of a single burn transfer

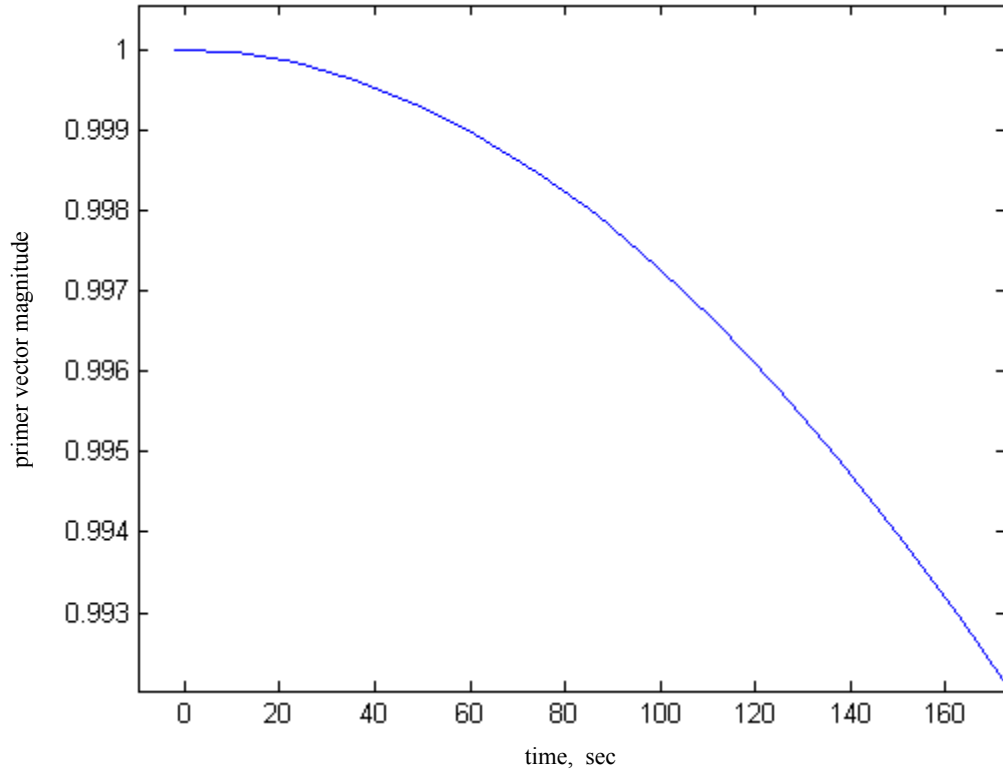


Fig. 3 Detail of the primer vector magnitude near the initial time

Table 4 Primer vector properties at t_r

primer vector angle with the velocity vector (deg)	0.01
primer rate vector angle with the radius vector (deg)	179.95

Conclusions

A method of targeting optimal low energy transfers, relying on multi-body dynamics, from an orbit about one body to another is presented. The targeting algorithm applies necessary conditions derived from primer vector theory to the orbit-to-capture transfer problem. With the help of precise derivatives obtained from the chain rule and the state transition matrix, optimal solutions can be converged using non-linear equation solving techniques.

The necessary conditions for an impulsive single impulse orbit-to-capture transfer offer a tool for evaluating low energy missions such as lunar tour missions of the outer planets or ballistic lunar capture trajectories. The use of primer vector theory to obtain necessary conditions for the problem offers an extension of previous work with primer vector theory. The resulting conditions can be seen as slight modifications of, but still consistent with,

previous results. For example, for the off boundary case, the primer vector of an optimal transfer must align with the velocity vector of the spacecraft with respect to the secondary mass, and the primer rate vector must align with the radius vector. Previously, it has been shown that the primer vector in an optimal orbit-to-orbit transfer must align with the impulse vector at the final time, and the time derivative of its magnitude must be zero. The new result is equivalent to previous findings so long as the second orbit insertion burn occurs at perigee and the burn is in the direction of the spacecraft's relative velocity, two conditions previously associated with optimal transfers. At the perigee point, the velocity vector is perpendicular to the radius vector. At this point on an optimal trajectory, both the radius vector and the primer vector have magnitudes that are unchanging.

Using the new necessary conditions, a ballistic lunar capture transfer was optimized and then analyzed to determine whether a second maneuver could improve the result. The transfer was found to be a local optimal with no available savings from an additional burn. It was therefore shown that a single burn transfer from the Earth to the Moon can meet the necessary conditions of optimality from primer vector theory.

References

- ¹ E.A. Belbruno, "Lunar Capture Orbits, A Method of Constructing Earth-Moon Trajectories and the Lunar GAS Mission," *Proceedings of AIAA/DGLR/JSASS Inter. Propl. Conf.*, AIAA Paper No. 97-1054, May 1987.
- ² E.A. Belbruno and J. Miller, "Sun-Perturbed Earth-to-Moon Transfers with Ballistic Capture," *J. Guid., Control and Dynamics*, Vol. 16, No. 4, pp. 770-775.
- ³ D. F. Lawden, *Optimal Trajectories for Spacecraft Navigation*, London, Butterworths, 1963.
- ⁴ P. M. Lion and M. Handelsman, "Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, vol. 6, 1968.
- ⁵ D. J. Jezewski, "Primer Vector Theory and Applications," NASA TR R-454, 1975.
- ⁶ J. E. Prussing, "Optimal Two- and Three-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal*, vol. 8, 1970.
- ⁷ L. R. Gross and J. E. Prussing, "Optimal Multiple-Impulse Direct Ascent Fixed-Time Rendezvous," *AIAA Journal*, vol. 12, 1974.
- ⁸ J. E. Prussing and J. H. Chiu, "Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits," *J. Guid., Control and Dynamics*, vol. 9. no. 1, 1986.
- ⁹ L. A. Hiday-Johnston and K. C. Howell, "Transfers Between Libration-Point Orbits in the Elliptic Restricted Problem," *Celestial Mechanics and Dynamical Astronomy*, vol 58, 1994.
- ¹⁰ L. A. D'Amario and T. N. Edelbaum, "Minimum Impulse Three-Body Trajectories," *AIAA Journal*, vol. 12, 1974.
- ¹¹ C. Ocampo, "Trajectory Optimization for Distant Earth Satellites and Satellite Constellations," PhD. Dissertation, University of Colorado at Boulder, 1996.
- ¹² C.L. Pu and T.N. Edelbaum, "Four-Body Trajectory Optimization," *AIAA Journal*, vol. 13, Mar. 1975, pp. 333-336.
- ¹³ D. G. Hull, *Optimal Control Theory for Applications*, New York, Springer-Verlag, 2003.

¹⁴ P. A. M Dirac, *Quantum Mechanics*, London, Oxford University Press, 1958.

¹⁵ “NS11AD”, Harwell Subroutine Library, URL: <http://hsl.rl.ac.uk/hsl2007/distrib/hsl2007.html>

¹⁶ E. M. Standish, “JPL Planetary and Lunar Ephemerides, DE405/LE405,” *Jet Propulsion Laboratory Interoffice Memorandum 312.F-98-048*, August 26, 1998.

¹⁷ R. H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, Virginia, American Institute of Aeronautics and Astronautics, Inc., 1999.

¹⁸ S. Zimmer and C. Ocampo, “Analytical Gradients for Gravity Assist Trajectories Using Constant Specific Impulse Engines,” *J. Guid., Control and Dynamics*, Vol. 28, No. 4, Jul. – Aug. 2005.

¹⁹ A.C. Hindmarsh, “Large Ordinary Differential Equation Systems and Software,” *IEEE Control Systems Magazine*, Vol. 2, No. 4, 1982, pp. 24-30.

²⁰ P. R. Griesemer and C. Ocampo, “An Efficient Strategy for Targeting Ballistic Lunar Capture Trajectories,” *Proceedings of AIAA/AAS Astrodynamics Conference*, Paper No. AAS 06-129, February 2006.